Projections and dilations on noncommutative L^p-spaces

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Projections and complemented subspaces

Let X be a Banach space. A projection is a bounded operator $P: X \to X$ such that $P^2 = P$.

A complemented subspace Y of X is the range of a bounded linear projection P. If the projection is contractive, we say that Y is contractively complemented.

Proposition

Let X be a Hilbert space. Then a subspace Y of X is contractively complemented if and only if Y is isometrically isomorphic to a Hilbert space.

Problem

To describe the contractively complemented subspaces of each Banach space X.

Projections on usual L^p-spaces

Theorem (Ando, Bernau, Douglas, Lacey, 1974 for the general measures)

Let Ω be a mesure space. Suppose $1 < p < \infty$ with $p \neq 2$.
For a subspace Y of $L^p(\Omega)$, the following statements are equivalent.

- *Y is the range of a positive contractive projection.*
- *Y is a closed sublattice of $L^p(\Omega)$.*
- *there exists a positive isometrical isomorphism from Y onto some L^p-space $L^p(\Omega')$.*

For a subspace Y of $L^p(\Omega)$, the following statements are equivalent.

- *Y is the range of a contractive projection.*
- *Y is isometrically isomorphic to some $L^p(\Omega')$.*

Projections on C^*-algebras and von Neumann algebras

Theorem (Effros, Ruan, 1974)

The range of any completely positive contractive projection $P: A \to A$ on a C^*-algebra A is complete order isometric to a C^*-algebra.

Theorem (Effros, Ruan, 1974)

The range of any normal completely positive contractive projection $P: M \to M$ on a von Neumann algebra M is $*$-isomorphic to a von Neumann algebra.

Contractives projections on Schatten spaces

The range of a contractive projection on the Schatten space S^p is not necessarily isometric to a Schatten space !
We let $\sigma \colon S^p \to S^p$ be the transpose map defined by

$$\sigma([x_{ij}]) = [x_{ji}].$$

We introduce the spaces

$$\mathrm{Sym}^p = \{x \in S^p \ : \ \sigma(x) = x\}$$

and

$$\mathrm{Asym}^p = \{x \in S^p \ : \ \sigma(x) = -x\}.$$

These subspaces are contractively complemented subspaces of S^p.
Indeed,

$$P_s = \frac{\mathrm{Id} + \sigma}{2} \quad \text{and} \quad P_a = \frac{\mathrm{Id} - \sigma}{2}$$

are contractive projections whose ranges are Sym^p and Asym^p.

Contractives projections on Schatten spaces

Arazy and Friedman have succeeded in establishing a complete classification of contractively complemented subspaces of S^p !

Theorem (Arazy, Friedman, 1992)

Suppose $1 \leq p < \infty$, $p \neq 2$. Let Y be a contractively complemented subspace of S^p.
Then Y is isometric to the ℓ^p-sum of subspaces of S^p, each of which is isometric to a subspace of S^p induced by a Cartan factor :

1. *a space of rectangular matrices*
2. *a space of anti-symmetric matrices*
3. *a space of symmetric matrices*
4. *a "spinorial space".*

Using Arazy-Friedman Theorem, we have :

Theorem (Le Merdy, Ricard, Roydor, 2009)

Suppose $1 \leq p < \infty$, $p \neq 2$. Let Y be a subspace of S^p. The following statements are equivalent.

- *The subspace Y is completely contractively complemented in S^p.*
- *There exist, for some countable set A, two families $(I_\alpha)_{\alpha \in A}$ and $(J_\alpha)_{\alpha \in A}$ of indices such that Y is completely isometric to the p-direct sum*

$$\bigoplus_{\alpha \in A}^{p} S^p_{I_\alpha, J_\alpha}.$$

Contractive projections on noncommutative L^1-spaces

Theorem (Friedman, Russo, 1985)

The range of contractive projection on the predual M_* of a von Neumann algebra M is isometric to the predual of a JW^*-triple, that is a weak-closed subspace of $B(H)$ closed under the triple product $xy^*z + zy^*x$.*

Theorem (Ng, Ozawa, 2002)

Let M be a von Neumann algebra. Let Y be a finite dimensional completely contractively complemented subspace of the predual M_* of M. Then Y is completely isometric to

$$S^1_{n_1,m_1} \oplus_1 \cdots \oplus_1 S^1_{n_k,m_k}$$

for some positive integers n_i, m_i.

Noncommutative L^p-spaces

Let $M \subset B(H)$ be a von Neumann algebra, i.e. a weak* closed involutive unital subalgebra of $B(H)$.

Suppose that M is equipped with a semifinite faithful normal trace $\tau \colon M_+ \to [0; \infty]$. Let S^+ be the set of all positive $x \in M$ such that $\tau(x) < \infty$ and S its linear span.

If $1 \leq p < \infty$, the non-commutative L^p-space $L^p(M) = L^p(M, \tau)$ is defined to be :

$$L^p(M) = \text{ completion of } \left\{ x \in S \ : \ \|x\|_{L^p(M)} = \tau((x^*x)^{\frac{p}{2}})^{\frac{1}{p}} \right\}.$$

We have $L^1(M) = M_*$. We let $L^\infty(M) = M$.

If $M = L^\infty(\Omega)$ and $\tau = \int_\Omega \cdot \, \mathrm{d}\mu$ we obtain $L^p(M) = L^p(\Omega)$.

If $M = B(\ell^2)$ and $\tau = \mathrm{Tr}$, we obtain $L^p(M) = S^p$.

Haagerup, Connes-Hilsum, Kosaki-Terp, Araki-Masuda... have given definitions of noncommutative L^p-spaces for a type III von Neumann algebra M equipped with a weight $\psi \colon M \to [0 + \infty]$.

Theorem (C. A., Y. Raynaud, 2015)

- *Suppose* $1 \leq p < \infty$.
- *Let* $P \colon L^p(M) \to L^p(M)$ *be a contractive completely positive projection.*

Then there exists a complete order isometrical isomorphism from the range of P *onto some noncommutative* L^p*-space* $L^p(N)$.

True for Haagerup L^p spaces.

Idea of the proof

Consider a projection $P\colon L^p(M) \to L^p(M)$.

First we consider the σ-finite case i.e. M equipped with a state φ.

Let $s(P)$ the supremum in M of the supports of the positive elements in $\mathrm{Ran}(P)$.

We begin to show that there exists a positive $h \in \mathrm{Ran}(P)$ such that

$$\mathrm{support}(h) = s(P).$$

We consider the restriction of P to

$$s(P)L^p(M)s(P) = L^p(s(P)Ms(P)).$$

We show this restriction is induced by a faithful normal conditional expectation.

For the non σ-finite case, we use some "covering and gluing argument".

Akcoglu Theorem

Theorem (Akcoglu, 1977)

- *Suppose* $1 \leq p < \infty$.
- *Let* $T\colon L^p(\Omega) \to L^p(\Omega)$ *is a positive contraction on* $L^p(\Omega)$.

Then there exists

- *another measure space* Ω',
- *an isometric embedding* $J\colon L^p(\Omega) \hookrightarrow L^p(\Omega')$ *and a contraction* $P\colon L^p(\Omega') \to L^p(\Omega)$,
- *an invertible isometry* $U\colon L^p(\Omega') \to L^p(\Omega')$

such that

$$T^n = PU^nJ, \qquad n \geq 0.$$

Dilation on noncommutative L^p-spaces

Definition (Le Merdy, Junge, 2007)

We say that a contraction $T\colon L^p(M) \to L^p(M)$ is dilatable if there exist

- *a noncommutative L^p-space $L^p(M')$,*
- *an isometric embedding $J\colon L^p(M) \to L^p(M')$ and a contractive map $P\colon L^p(M') \to L^p(M)$,*
- *an invertible isometry $U\colon L^p(M') \to L^p(M')$*

such that

$$T^n = PU^nJ, \qquad n \geq 0.$$

In this context, Akcoglu Theorem has no noncommutative analog for completely positive contractions on Schatten spaces S^p (Junge, Le Merdy, 2007).

However, a lot of contractive operators on noncommutative L^p-spaces admits some dilations : some Schur multipliers $M_A\colon S^p \to S^p$ and some Fourier multipliers...

Strongly continuous semigroups

Definition

A strongly continuous semigroup (or C_0-semigroup) on a Banach space X is a family of operators $(T_t)_{t \geq 0}$ where $T_t \colon X \to X$ such that :

$$T_0 = I, \quad \text{and} \quad T_{t+s} = T_t T_s, \qquad t, s \geq 0$$

with

$$t \mapsto T_t x$$

continuous for any $x \in X$.

Fendler Theorem

Fendler showed a continuous version of Akcoglu theorem :

Theorem (Fendler, 1997)

- *Suppose* $1 < p < \infty$.
- *Let* $(T_t)_{t \geq 0}$ *be a strongly continuous semigroup of positive contractions acting on* $L^p(\Omega)$.

Then there exists

- *a measure space* Ω',
- *a strongly continuous group of invertible isometries* $(U_t)_{t \geq 0}$ *acting on* $L^p(\Omega')$,
- *an isometric embedding* $J \colon L^p(\Omega) \hookrightarrow L^p(\Omega')$ *and a contractive map* $P \colon L^p(\Omega') \to L^p(\Omega)$

such that

$$T_t = PU_tJ, \qquad t \geq 0.$$

Dilation of semigroups on noncommutative L^p-spaces

Theorem (C. A., Y. Raynaud, 2015)

- *Suppose* $1 < p < \infty$.
- *Let* M *be a von Neumann algebra equipped with a state.*
- *Let* $(T_t)_{t \geq 0}$ *be a* C_0*-semigroup of completely positive contractions on* $L^p(M)$.
- *Suppose that each* $T_t \colon L^p(M) \to L^p(M)$ *is dilatable.*

Then there exists

- *a noncommutative* L^p*-space* $L^p(M')$,
- *a strongly continuous group of isometries* $U_t \colon L^p(M') \to L^p(M')$,
- *an isometric embedding* $J \colon L^p(M) \to L^p(M')$ *and a contractive map* $P \colon L^p(M') \to L^p(M)$

such that

$$T_t = PU_tJ, \qquad t \geq 0.$$

Idea of the proof

We have a dilation of $T_{\frac{1}{n}}$:

$$(T_{\frac{1}{n}})^k = P_{\frac{1}{n}}(U_{\frac{1}{n}})^k J_{\frac{1}{n}}, \qquad k \geq 0.$$

For a finite set $B \subset \mathbb{Q}^+$ let $V_B = \{n \in \mathbb{N} : \; nt \in \mathbb{N} \text{ for any } t \in B\}$. The set of all sets V_B ($B \subset \mathbb{Q}^+$, B finite) is the basis of some filter which is contained in some ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

For $t \in \mathbb{Q}^+$, we define the operator $U_{n,t} \colon L^p(N_{\frac{1}{n}}) \to L^p(N_{\frac{1}{n}})$ by

$$U_{n,t} = \begin{cases} (U_{\frac{1}{n}})^{nt} & \text{if} \quad nt \in \mathbb{N} \\ \mathrm{Id}_{L^p(N_{\frac{1}{n}})} & \text{if} \quad nt \notin \mathbb{N}. \end{cases}$$

For any finite subset B of $\mathbb{Q}^+$, any $t \in B$ and any $n \in V_B$, we obtain :

$$T_t = (T_{\frac{1}{n}})^{nt} = P_{\frac{1}{n}} U_{n,t} J_{\frac{1}{n}}.$$

Idea of the proof

We consider the ultraproduct

$$\prod_{\mathcal{U}} L^p(N_{\frac{1}{n}}) = L^p(\tilde{N})$$

for some von Neumann algebra $\tilde{N}$.

If $\mathcal{I}\colon L^p(M) \to L^p(M)^{\mathcal{U}}$, $x \mapsto (x, x, \ldots)^{\bullet}$ is the canonical map and, by letting

$$\widetilde{J} = \Big(\prod_{\mathcal{U}} J_{\frac{1}{n}}\Big)\mathcal{I}, \qquad \widetilde{P} = Q \prod_{\mathcal{U}} P_{\frac{1}{n}}, \qquad \widetilde{U}_t = \prod_{\mathcal{U}} U_{n,t}, \quad t \in \mathbb{Q}^+,$$

where $Q\colon L^p(M)^{\mathcal{U}} \to L^p(M)$ is the canonical projection we obtain a dilation

$$T_t = \tilde{P}\tilde{U}_t\tilde{J}, \qquad t \in \mathbb{Q}^+.$$

on a "big" noncommutative L^p-space $L^p(\tilde{N})$.

We need to extend this dilation by continuity to obtain some $(\tilde{U}_t)_{t \geq 0}$.

But the ultraproduct may be to large for the map $\mathbb{Q}^+ \to L^p(\tilde{N})$, $t \mapsto \tilde{U}_t x$ to be continuous if $x \in L^p(\tilde{N})$!

Idea of the proof

We have

$$T_t = \tilde{P}\tilde{U}_t\tilde{J}, \qquad t \in \mathbb{Q}^+.$$

We construct some completely positive contractive projection Q from $L^p(\tilde{M})$ on the subspace of "continuously translating elements"

$$L^p(\tilde{M})_c = \{x \in L^p(\tilde{M}) \ : \ s \mapsto \tilde{U}_t x \text{ is continuous on } \mathbb{Q}^+\}$$

using some tricks from de Leeuw and Glicksberg (1965).

This space is a noncommutative L^p-space by the result on projections.

With this projection, we can restrict the dilation on a smaller space since we can show that $\tilde{J}(L^p(M)) \subset L^p(\tilde{M})_c$.

Finally we obtain a dilation

$$T_t = PU_tJ, \qquad t \in \mathbb{R}.$$

with $U_t \colon L^p(\tilde{M})_c \to L^p(\tilde{M})_c$.

Markov maps

Definition

Let (M, ϕ) and (N, ψ) be von Neumann algebras equipped with normal faithful states ϕ and ψ respectively.
A linear map $T \colon M \to N$ is called a (ϕ, ψ)-Markov map if

- *T is unital and completely positive,*
- *$\psi \circ T = \phi$,*
- *$T \circ \sigma_t^\phi = \sigma_t^\psi \circ T$, for all $t \in \mathbb{R}$, where $(\sigma_t^\phi)_{t \in \mathbb{R}}$ and $(\sigma_t^\psi)_{t \in \mathbb{R}}$ denote the automorphism groups of the states ϕ and ψ respectively.*

When $(M, \phi) = (N, \psi)$, we say that T is a ϕ-Markov map.

It is known that there exists a unique completely positive, unital map $T^* \colon N \to M$ such that

$$\phi(T^*(y)x) = \psi(yT(x)), \qquad x \in M, \ y \in N.$$

We say that T is selfadjoint if $T^* = T$.

Factorizable maps

Definition (Anantharaman-Delaroche, 2006)

A (ϕ,ψ)-Markov map $T\colon M \to M$ is called factorizable if there exist

- *a von Neumann algebra P equipped with a faithful normal state χ,*
- *and some $*$-monomorphisms $J_0\colon M \to P$ and $J_1\colon M \to P$,*

such that J_0 is (ϕ,χ)-Markov and J_1 is (ψ,χ)-Markov satisfying

$$T = J_0^* \circ J_1.$$

If T is factorizable then we have a "dilation" : there exists

- a von Neumann algebra M' with a normal faithful state ψ,
- an automorphism U of M' leaving ψ invariant,
- a (ϕ,ψ)-Markov $*$-monomorphism $J\colon M \to M'$

satisfying

$$T^n = J^* \circ U^n \circ J, \quad n \geq 0.$$

Here $J^*\colon M' \to M$ is a conditional expectation (Haagerup-Musat).

Examples of factorizable maps

- Selfadjoint unital completely positive Schur multipliers

$$M_A \colon B(\ell^2) \to B(\ell^2)$$

are factorizable Tr-Markov maps (Ricard, 2008).
Here "selfadjoint" translates into "defined by a real matrix A".

- Selfadjoint unital completely positive Fourier multipliers

$$M_\varphi \colon \mathrm{VN}(G) \to \mathrm{VN}(G)$$

on a discrete group G are factorizable τ_G-Markov maps (Ricard, 2008).
Here "selfadjoint" translates into "defined by a real function $\varphi \colon G \to \mathbb{R}$".

- Completely positive second quantization operators

$$\Gamma_q(T) \colon \Gamma_q(H) \to \Gamma_q(H)$$

are factorizable if T is selfadjoint (in the usual sense !).

Dilations of semigroups

Definition

- *Let M be a von Neumann algebra equipped with a normal faithful state ϕ.*
- *Let $(T_t)_{t \geq 0}$ be a w^*-continuous semigroup of ϕ-Markov maps on M.*

We say that the semigroup is dilatable if there exist

- *a von Neumann algebra M' equipped with a normal faithful state ψ,*
- *a w*-continuous group $(U_t)_{t \in \mathbb{R}}$ of ϕ-Markov $*$-automorphisms of M',*
- *a (ϕ, ψ)-Markov $*$-monomorphism $J \colon M \to M'$*

such that

$$T_t = \mathbb{E} \circ U_t \circ J, \qquad t \geq 0,$$

where $\mathbb{E} = J^* \colon M' \to M$ is the canonical faithful normal conditional expectation preserving the states associated with J.

Dilation of semigroups on von Neumann algebras

Using a similar ultraproduct approach to the case $p < \infty$, we obtain :

Theorem (C.A., Y. Raynaud, 2015)

- *Let M be a von Neumann algebra equipped with a normal faithful state ϕ.*
- *Let $(T_t)_{t \geq 0}$ be a w^*-semigroup of factorizable ϕ-Markov maps on M.*

Then the semigroup $(T_t)_{t \geq 0}$ is dilatable.

Junge, Ricard and Shlyakhtenko announced the same result for finite traces for selfadjoint ϕ-Markov maps but no assumption of factorizability using different methods.

Dilations of semigroups of Fourier multipliers

Our result gives a short proof of the following result :

Corollary

- *Let G be a discrete group and $\mathrm{VN}(G)$ be its von Neumann algebra equipped with its trace τ_G.*
- *Let $(T_t)_{t\geq 0}$ be a w^*-semigroup of completely positive unital self-adjoint Fourier multipliers on $\mathrm{VN}(G)$.*

Then the semigroup $(T_t)_{t\geq 0}$ is dilatable.

We also have a similar result for semigroups of Schur multipliers (already known with an elementary construction, C.A. 2010)

Schur multipliers

An operator in $B(\ell^2)$ can be seen as a matrix with the canonical basis of ℓ^2.

A Schur multiplier defined by a matrix A is a linear map

$$\begin{array}{rccc} M_A: & B(\ell^2) & \longrightarrow & B(\ell^2) \\ & [x_{ij}] & \longmapsto & [a_{ij}x_{ij}]. \end{array}$$

Recall that

$$S^p = \left\{ x \in B(\ell^2) \colon \|x\|_{S^p} = \left(\mathrm{Tr}\,|x|^p\right)^{\frac{1}{p}} < \infty \right\}$$

where $|x| = (x^*x)^{\frac{1}{2}}$.

Contractive Schur multipliers

Theorem (Grothendieck, 1956)

Let $M_A\colon B(\ell^2) \to B(\ell^2)$ be a Schur multiplier defined by a matrix A. It is contractive if and only if there exist

- *a Hilbert space H*
- *two sequences of vectors $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ of H of norm ≤ 1*

such that

$$a_{ij} = \langle \alpha_i, \beta_j \rangle_H.$$

Moreover if M_A is selfadjoint (i.e. selfadjoint on S^2), we can take a real hilbert space H.
If M_A is positive, we can take $\alpha_i = \beta_i$.

Semigroups of Schur multipliers

Theorem (C. A., 2014)

The w^*-semigroups of contractive Schur multipliers on $B(\ell^2)$ which are selfadjoint on S^2 are precisely the semigroups

$$T_t = \left[\mathrm{e}^{-t\|\alpha_i-\beta_j\|_H^2}\right]_{i,j\geq 1}$$

where α_i, β_j are elements of a real Hilbert space H.
If each T_t is positive, we can take $\alpha_i = \beta_i$.

This theorem is a continuous analog of Grothendieck Theorem.

The proof uses again ultraproducts and the discrete case !

Ultraproducts are a bridge between discrete analysis and continuous analysis.

It was a pleasure to present this work to you !