Quantum Capacity for nice channels

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joint work with Li Gao and Nicholas LaRacurente

Overview

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- Classical Entropy and capacity

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- Quantum channels and capacities

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- Estimates for (quantum) group channels

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- Tools

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- The minimal entropy $H_{min}(\Phi) = \min_p H(\Phi(p)))$ is a measure of how noisy a channel is. $H_{\min}(id) = 0$

Relative Entropy and mutual information

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- Let p^{XY} be a distribution on a product space and $p^X(x) = \sum_y p(x,y)$, $p^Y = \sum_x p(x,y)$ be the marginals, $p(x|y) = p(x,y)/p^Y(y)$ the conditional probability.

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- and

$$I(X,Y)_p = H(X)+H(Y)-H(XY) \ = \ H(p^X)+H(p^Y)-H(p^{XY})$$

  is the mutual information.

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- For a channel $\mathcal{N} : \ell_1^m \to \ell_1^m$ the capacity is the rate rn of faithfully transmitted bits for n uses of the channel

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- $H_{\min} = \lim_{p \to \infty} p \log \|\mathcal{N}^* : \ell_p \to \ell_\infty\|$.

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But only for classical channels!

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- Hastings (2008) showed that

$$H_{\min}(\Phi \otimes \Psi) \neq H_{\min}(\Phi) + H_{\min}(\Psi)$$

in general.

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- (JP) $C^{(d)}$ is in general not additive. Schor considered related restriction.
- For $d = 1$ we find Holevo's capacity. Shor showed (before Hastings) that the additivity of the min-entropy is equivalent to the additive of the one shot Holevo capacity.

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- For $d = \dim(H_{A'}) = |A'|$ we find the classical capacity with assisted entanglement

$$C_{EA}(\mathcal{N}) \;=\; \sup_{\rho^{A'A}} I(A,B)_{id \otimes \mathcal{N}(\rho^{A'A})}$$

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- The classical capacity with assisted entanglement is a rate (early succes in QIT: HSW, Schor, Devetak, Winter,...)

Coherent information

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- Let $\mathcal{N} : S_1(A') \to S_1(B)$ be a channel and $\rho^{AB} = id \otimes \mathcal{N}(\rho^{A'A})$ the image of a bipartite state. We may write the mutual information as

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- Indeed, according to [DJKRS]

$$I_c(A\rangle B) = c\frac{d}{dp}\|\rho^{BA}\|_{S_1(H_B;S_p(H_A))}|_{p=1}$$

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- Using the contraction $id \otimes tr_C : S_1(H_{BC}; S_p(H_A)) \to S_1(H_B; S_p(H_A))$, we deduce by differentiation strong super additivity

$$H(B) - H(AB) \le H(BC) - H(ABC)\;.$$

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gives the rate R of transmitting rn qubits per n uses of the channel
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- The regularization is general hard to calculate. By teleportation we have $2Q(\mathcal{N}) \le 2Q_E(\mathcal{N}) \;=\; C_{EA}(\mathcal{N})$.

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- Our main result is that there is a nice class of channels $\theta_f : S_1(L_2(M)) \to S_1(L_2(M))$ such that

$$\begin{aligned} \max\{\max_k \ln n_k, \tau(f \ln f)\} &\leq Q(\theta_f) \leq Q^{pot}(\theta_f) \\ &\leq \max_k \ln n_k + \tau(f \ln f) \ . \end{aligned}$$

Here $f \in N$ is the symbol of a channel and $M = \oplus_k M_{n_k}$ is a finite dimensional von Neumann algebra.

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  Here $f \in N$ is the symbol of a channel and $M = \oplus_k M_{n_k}$ is a finite dimensional von Neumann algebra.
- For these channels we also have estimates for the (CQE) region (classical, quantum, entanglement) region of the channel, which goes back to Schor, see also Wilde.

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❄ Let G be a finite group, $\lambda(g)e_h = e_{gh}$ the left regular representation on $\ell_2(G) = L_2(L(G))$. Let $f : G \to \mathbb{R}_+$ and

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- Then θ_1 is the conditional expectation onto right regular representation $L(G)' = R(G) = \sum_k M_{n_k} = M$. We have

$$\max(\max_k \ln n_k, \tau(f\ln f)) \le Q^{(1)}(\theta_f) \;\le\; \tau(f\ln f) + \max_k \ln n_k\;.$$

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- Let G be a finite group, $\lambda(g)e_h = e_{gh}$ the left regular representation on $\ell_2(G) = L_2(L(G))$. Let $f : G \to \mathbb{R}_+$ and

$$\theta_f(\rho) \;=\; \frac{1}{|G|}\sum_{g\in G} f(g)\lambda(g)^*\rho\lambda(g)\;.$$

- Then θ_1 is the conditional expectation onto right regular representation $L(G)' = R(G) = \sum_k M_{n_k} = M$. We have

$$\max(\max_k \ln n_k, \tau(f\ln f)) \le Q^{(1)}(\theta_f) \;\le\; \tau(f\ln f) + \max_k \ln n_k\;.$$

  Both lower bounds are achievable as well as the upper bound for $G = \mathbb{Z}_n$.
- For $f = 1$ we find the conditional expection onto $R(G)$. $R(G)$ belongs to the fixpoint algebra of *every* θ_f, this gives one of the lower estimates.

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- For the conditional expectation $\theta_f = \theta_1$ we have equality $Q(\theta_f) = \max_k \ln n_k$. Teleportation $Q(\theta_1) \leq 2C_{EA}(\theta_1) = \ln n$ confirms that for groups the maximal dimension of an irreducible representation π satisfies

$$|\dim(\pi)| \ \leq \ |G|^{1/2} \ .$$

- According to Vershik et all, the symmetric group $n = m!$ satisfies $\max_k \ln n_k \ \geq \ \sqrt{n} - c\sqrt{m}$ and hence we find examples where entaglement improves the rate only by a very small amount $\sqrt{m}$ which is small compared to the value $\sqrt{m!}$.

- For $G = \mathbb{Z}_d^m \rtimes \mathbb{Z}_m$ the maximal dimension of an irreducible representation m is small compared to $n = |G| = md^m$ and hence our new estimates outperform the more classical upper estimates

$$C_{EA}(\theta_f) \; = \; \ln n + \tau(f \ln f)$$

Coincidentally, this estimate is also new (but related to previous work with Ruan and Neufang).

Schur multipliers

- Starting from a finite group we can also define the Schur multiplier

$$\theta_f([x_{gh}]) \;=\; \sum_{gh} f(g^{-1}h)x_{gh}$$

given by pointwise multiplication of the matrix $x = [x_{gh}]$. We may assume that $f(g) = \tau(d\lambda(g))$ given by the normalized trace on $L(G)$.

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- It was independtly proved by Neufang and Crann that
$$Q(\theta_f) \;=\; \tau(f \ln f)\,.$$
- Schur mutlipliers are so-called degradable channels and those don't require regularization! Our previous group channels, however, are in general not degradable.

❄ Winter et al introduced the potential capacity

$$Q^{(pot)}(\mathcal{N}) \;=\; \sup_{\mathcal{N}'} Q^{(1)}(\mathcal{N}\otimes\mathcal{N}') - Q^{(1)}(\mathcal{N}) \, .$$

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- Our upper estimates also hold for $Q^{(pot)}$.

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- Our class of channels include quantum group channels, in particular Drinfeld doubles, which have Kraus operators which are neither unitaries or projections.

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- Our class of channels include quantum group channels, in particular Drinfeld doubles, which have Kraus operators which are neither unitaries or projections.
- Our examples also include random unitaries given by products of the (Majorana) Clifford generators. However, in these particular cases only the estimates for $Q^{(pot)}$ are ‘new’. As of now better estimates are obtained by just adding entanglement and use teleportation.

Tools

- (Stinespring-Kraus-Haydon/Winter) Every channel comes with a partial isometry
$$V : H_A \to H_B \otimes H_E$$
and hence a subspace $\mathrm{st}(\mathcal{N}) \subset H_B \otimes H_E$ of a tensor product of two Hilbert spaces.

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  where $R_d(X) = X \otimes_h R_d$ and $C_p^d(X) = C_p \otimes_h X$. Indeed,

$$Q^{(1)}(\mathcal{N}) \; = \; \sup_d \frac{d}{dp} \mathrm{rc}_{\mathrm{p,d}}(\mathrm{st}_p(\mathcal{N}))^2 \, .$$

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- Using complex interpolation we show that

$$\|id \otimes \theta_f(\rho)\|_p \ \le \ \|f\|_p \|id \otimes \theta_1(\rho)\|_p \ \le \ \|f\|_p \|id \otimes \theta_f(\rho)\|_p \ .$$

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- Using similar techniques we will prove

$$\begin{aligned} \|\theta_f \otimes id(\rho)\|_{S_1(H_B;S_p(H_A))} &\le \|f\|_p \|\theta_1 \otimes id(\rho)\|_{S_1(H_B;S_p(H_A))} \\ &\le \|f\|_p \|\theta_f \otimes id(\rho)\|_{S_1(H_B;S_p(H_A))} \ . \end{aligned}$$

  This allows us to prove estimates for the private capacity.

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- Schur mutlipliers $\mathcal{N}$ are degradable, i.e. there exists a channel Ψ such that $\mathcal{N}^c = \Psi\mathcal{N}$ holds for the complementary channel (trace out Bob instead of environment).
- An important tool for our analysis feature is the standard form of a finite von Neumann algebra.

Thanks for listening