Decomposable Schur multipliers and non-commutative Fourier multipliers

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Regular operators on classical $L^p(\Omega)$ spaces

Let $1 \leq p \leq \infty$, and (Ω_k, μ_k) be two σ-finite measure spaces $(k = 1, 2)$. An operator $T : L^p(\Omega_1) \to L^p(\Omega_2)$ is called positive if for $f \in L^p(\Omega_1)$, $f \geq 0$ pointwise, we always have $Tf \geq 0$ pointwise.

An operator $T : L^p(\Omega_1) \to L^p(\Omega_2)$ is called regular if $T = T_1 - T_2 + i(T_3 - T_4)$ with T_1, T_2, T_3, T_4 positive operators.

THEOREM: Let $T : L^p(\Omega_1) \to L^p(\Omega_2)$ be a regular operator, X a Banach space and $S : X \to X$ a bounded operator. Then the tensor product $T \otimes S : L^p(\Omega_1) \otimes X \subset L^p(\Omega_1; X) \to L^p(\Omega_2; X)$ extends to a bounded operator on the Bochner space $L^p(\Omega_1; X)$ with $\|T \otimes S\| \leq \|T\|_{reg} \|S\|$. Here,
$\|T\|_{reg} = \sup_{n\in\mathbb{N}} \|T \otimes I_{\ell_n^\infty}\|_{L^p(\Omega_1;\ell_n^\infty)\to L^p(\Omega_2;\ell_n^\infty)} < \infty.$

Schatten classes and non-commutative $L^p(M)$ spaces

Let I be a non-empty index set and $1 \leq p < \infty$. Then the **Schatten class** S_I^p is defined to be the class of all compact operators T on ℓ_I^2 such that $\mathrm{tr}((T^*T)^{p/2}) < \infty$.
$S_I^\infty = \{\text{compact operators on } \ell_I^2\}$.

Let $M \subset B(H)$ be a von Neumann algebra, i.e. weak* closed involutive subalgebra of $B(H)$. Assume that M is equipped with a semifinite faithful normal trace $\tau : M_+ \to [0, \infty]$. Then for $1 \leq p < \infty$, the **non-commutative L^p space** is defined to be:
$L^p(M) = L^p(M, \tau) =$ completion of
$\{x \in M : \|x\|_{L^p(M)} = \tau((x^*x)^{p/2})^{\frac{1}{p}} < \infty\}$.
$L^\infty(M) := M$.

For example, $L^p(\Omega) = L^p(L^\infty(\Omega), \int_\Omega \cdot d\mu)$, and $S_I^p = L^p(B(\ell_I^2), \mathrm{tr})$ for $1 \leq p < \infty$.

Completely bounded and completely positive mappings

S_I^p and $L^p(M)$ are Banach spaces, but even more:
Let $n \in \mathbb{N}$. Define a norm on $M_n \otimes L^p(M) = M_n(L^p(M))$ by [Pisier]

$$\|[x_{ij}]\|_{M_n(L^p(M))} = \sup\{\|\alpha \cdot x \cdot \beta\|_{L^p(M_n(M))} : \|\alpha\|_{S_n^{2p}}, \|\beta\|_{S_n^{2p}} \leq 1\}.$$

$L^p(M)$ is called an **operator space** equipped with the sequence of norms on $M_n(L^p(M))$, $n \in \mathbb{N}$.
A mapping $u : L^p(M_1) \to L^p(M_2)$ is called **completely bounded** if the family of mappings

$$u_n : M_n(L^p(M_1)) \to M_n(L^p(M_2)), \; [x_{ij}] \mapsto [u(x_{ij})]$$

satisfy $\|u\|_{cb} = \sup_{n\in\mathbb{N}} \|u_n\| < \infty$.
Further, u is called **completely positive**, if all the mappings u_n are positive, where $x \in M_n(L^p(M_1))$ is defined to be positive if $x = y^* y$ with $y \in M_n(L^{2p}(M_1))$.

Completely positive mappings and classical $L^p(\Omega)$ spaces

PROPOSITION: Let $1 \leq p \leq \infty$. Let $L^p(\Omega)$ be a classical L^p space and $L^p(M)$ a non-commutative one. Then a positive mapping $u : L^p(M) \to L^p(\Omega)$ is completely positive.

Idea of proof: Let $a \in M_{n,1}$ and $[x_{ij}] \in M_n(L^p(M))$ positive. Then $a^*[x_{ij}]a \in L^p(M)$ is positive, hence $u(a^*[x_{ij}]a) \in L^p(\Omega)$ is positive. Thus,

$$a^*[u(x_{ij})(\omega)]a = \sum_{i,j=1}^{n} \overline{a_i} u(x_{ij})(\omega) a_j = u(\sum_{i,j=1}^{n} \overline{a_i} x_{ij} a_j)(\omega) = u(a^*[x_{ij}]a)(\omega)$$
$$\geq 0.$$

Then for a.a. $\omega \in \Omega$, $[u(x_{ij})(\omega)]$ is positive in M_n. Now use the fact that $[f_{ij}] \in M_n(L^p(\Omega))$ is positive if and only if $[f_{ij}(\omega)] \in M_n$ is a positive matrix for almost all $\omega \in \Omega$. Hence $[u(x_{ij})]$ is positive in $M_n(L^p(\Omega))$.

COROLLARY: Any positive mapping $u : L^p(\Omega) \to L^p(M)$ is completely positive.

Definition of decomposable mappings

DEFINTION: Let $1 \leq p \leq \infty$ and $T : L^p(M_1) \to L^p(M_2)$ be a bounded linear mapping. Then T is called **decomposable** if $T = T_1 - T_2 + i(T_3 - T_4)$ with completely positive mappings T_1, T_2, T_3, T_4. The set of decomposable operators $\mathrm{Dec}(L^p(M_1), L^p(M_2))$ is a Banach space equipped with the norm

$$\|T\|_d = \sup_{|\lambda|\leq 1} \inf\{\|T_1\| + \|T_2\| + \|T_3\| + \|T_4\| : \lambda T = T_1 - T_2 + i(T_3 - T_4)\}.$$

Equivalent norm for decomposable mappings

PROPOSITION: Let $1 \leq p \leq \infty$ and let $T : L^p(M_1) \to L^p(M_2)$ be a bounded linear mapping. Then T is decomposable if and only if there exist $v_1, v_2 : L^p(M_1) \to L^p(M_2)$ such that the mapping

$$\begin{pmatrix} v_1 & T \\ T^\circ & v_2 \end{pmatrix} : S_2^p(L^p(M_1)) \to S_2^p(L^p(M_2))$$

is completely positive, where $T^\circ(x) = (T(x^*))^*$. We let $\|T\|_{dec} = \inf\{\max\{\|v_1\|_{cb}, \|v_2\|_{cb}\}\}$, where the infimum runs over all possible v_1, v_2. Then $\|T\|_d$ and $\|T\|_{dec}$ are equivalent on $\mathrm{Dec}(L^p(M_1), L^p(M_2))$.

Properties of decomposable mappings

PROPOSITION: Let M_1, M_2 be QWEP von Neumann algebras. Let $1 < p < \infty$. Then any decomposable map $T : L^p(M_1) \to L^p(M_2)$ is completely bounded and $\|T\|_{cb} \leq \|T\|_{dec}$. In particular, any completely positive mapping $T : L^p(M_1) \to L^p(M_2)$ is completely bounded.

THEOREM [Pisier]: Let M_1, M_2 be hyperfinite von Neumann algebras. Then $T : L^p(M_1) \to L^p(M_2)$ is decomposable if and only if for any operator space E, $T \otimes I_E : L^p(M_1; E) \to L^p(M_2; E)$ is bounded. In this case, in fact $\|T \otimes I_E\| \leq C\|T\|_{dec} < \infty$, and $\sup_{n\in\mathbb{N}} \|T \otimes I_{M_n}\|_{L^p(M_1;M_n)\to L^p(M_2;M_n)} \cong \|T\|_d \cong \|T\|_{dec}$.

Decomposable vs. completely bounded mappings

PROPOSITION [Haagerup $p=\infty$, A.-K.]: Let M have a finite trace τ and $u_1,\ldots,u_n \in M$ be arbitrary unitaries. Let $1 \le p \le \infty$. Consider the map $T : \ell_n^p \to L^p(M)$ defined by $T(e_k) = u_k$. Then $\|T\|_{dec} = n^{1-\frac{1}{p}}$.

Consider now $\mathbb{F}_n$ the free group of n generators $g_1, g_2, \ldots, g_n$, and $VN(\mathbb{F}_n)$ the **group von Neumann algebra**, contained in $B(\ell^2(\mathbb{F}_n))$, generated by the unitary elements $\lambda_s(f) = f(s^{-1}\cdot)$.

THEOREM [Haagerup $p=\infty$, A.-K.]: Let $1 \le p \le \infty$. Let $n \ge 2$ be an integer. The map $T_n : \ell_n^p \to L^p(VN(\mathbb{F}_n))$, $e_k \mapsto \lambda_{g_k}$ satisfies $\|T_n\|_{cb} \le (2\sqrt{n-1})^{1-\frac{1}{p}}$ and $\|T_n\|_{dec} = n^{1-\frac{1}{p}}$. In particular, $\|T_n\|_{dec}/\|T_n\|_{cb} \to \infty$ as $n \to \infty$.

Open questions

Question 1: Let R be the hyperfinite factor of type II_1 and let $U_1, \ldots, U_n \in R$ be a sequence of self-adjoint anticommuting operators. Suppose $1 \leq p \leq \infty$. Consider the map $T : \ell_n^p \to L^p(R)$ defined by $T(e_k) = U_k$.
What are the values of $\|T\|$, $\|T\|_{dec}$, $\|T\|_{cb}$?

Question 2: Let $1 \leq p \leq \infty$. Do we have for every map $T : \ell_2^p \to L^p(M)$ the equalities $\|T\| = \|T\|_{cb} = \|T\|_{dec}$?
True for $p = \infty$ [Haagerup].

Question 3: Let $1 \leq p \leq \infty$. Suppose that for every map $T : \ell_3^p \to L^p(M)$ we have $\|T\| = \|T\|_{cb} = \|T\|_{dec}$.
Is M necessarily hyperfinite?
Even open for $p = \infty$.

Definition of Schur multipliers

Let I be some index set, $1 \leq p \leq \infty$, and $\phi : I \times I \to \mathbb{C}$ be a bounded function. A mapping $M_\phi : S_I^p \to S_I^p$ is called S^p-**Schur multiplier** if it is of the form $M_\phi([x_{ij}]) = [\phi(i,j)x_{ij}]$.

Complementation of Schur multipliers

THEOREM [A.-K.]: Let I be some index set. For a completely bounded mapping $S : S_I^p \to S_I^p$ let $\phi_S : I \times I \to \mathbb{C}$, $(i, j) \mapsto \mathrm{tr}(S(e_{ij})e_{ji})$. Then the linear mapping

$$P_I : CB(S_I^p) \to CB(S_I^p),\ S \mapsto M_{\phi_S}$$

has the following properties:

1. P_I takes its values in the completely bounded S^p-Schur multipliers.
2. P_I is contractive.
3. $P_I(S) = S$ as soon as S is already a cb S^p-Schur multiplier.
4. $P_I(S)$ is completely positive as soon as S is completely positive.

Proof of Complementation of Schur multipliers

PROOF: Let $\Delta : B(\ell^2_I) \to B(\ell^2_I)\overline{\otimes}B(\ell^2_I)$ be the normal $*$-isomorphism which preserves the traces onto the sub von Neumann algebra $\Delta(B(\ell^2_I)) \subseteq B(\ell^2_I)\overline{\otimes}B(\ell^2_I)$ such that

$$\Delta(e_{ij}) = e_{ij} \otimes e_{ij}, \quad (i, j \in I).$$

Let $\mathbb{E}$ be the normal conditional expectation of $B(\ell^2_I)\overline{\otimes}B(\ell^2_I)$ onto $\Delta(B(\ell^2_I))$ that leaves $\mathrm{tr} \otimes \mathrm{tr}$ invariant. For any $i, j, k, l \in I$ we have $\mathbb{E}(e_{ij} \otimes e_{kl}) = \delta_{ik}\delta_{jl}e_{ij} \otimes e_{ij}$. Set now $P_I(S) = \Delta^{-1}\mathbb{E}(S \otimes \mathrm{Id}_{S^p_I})\Delta$. If S completely positive, then also $P_I(S)$ is. Moreover,

$$\begin{aligned}\|P_I(S)\|_{cb,S^p_I \to S^p_I} &\leq \|\Delta^{-1}\mathbb{E}(S \otimes \mathrm{Id}_{S^p_I})\Delta\|_{cb} \\ &\leq \|S\|_{cb,S^p_I \to S^p_I}.\end{aligned}$$

Finally check that $P_I(S)$ is a Schur multiplier and $P_I(S) = S$ if S is already a Schur multiplier.

Consequences of the complementation

COROLLARY: Let I be an index set, $1 < p < \infty$ and $\phi : I \times I \to \mathbb{C}$ a bounded function. Then M_ϕ is a **decomposable S^p-Schur multiplier** if and only if M_ϕ is a **bounded Schur multiplier** $B(\ell^2_I) \to B(\ell^2_I)$.

Proof: "$\Longrightarrow$": Let $M_\phi : S^p_I \to S^p_I$ be decomposable. Then $M_\phi = R_1 - R_2 + i(R_3 - R_4)$ with completely positive R_k. Thus, $M_\phi = P_I(M_\phi) = P_I(R_1) - P_I(R_2) + i(P_I(R_3) - P_I(R_4))$. Now each $P_I(R_k)$ is a completely positive S^p-Schur multiplier, which is known to be bounded on $B(\ell^2_I)$. Thus, also $M_\phi = P_I(M_\phi)$ is bounded on $B(\ell^2_I)$.
"$\Longleftarrow$": M_ϕ bounded on $B(\ell^2_I) \Longrightarrow$ completely bounded on $B(\ell^2_I) \Longrightarrow$ decomposable on $B(\ell^2_I) \Longrightarrow$ decomposable on S^p_I.

Strongly non decomposable operators

DEFINITION ([Arendt-Voigt 1991] in the case of Fourier multipliers on abelian groups): Let $T : L^p(M_1) \to L^p(M_2)$ be completely bounded. T is called **CB strongly non decomposable** if T does not belong to $\overline{\text{Dec}(L^p(M_1), L^p(M_2))}$, closure in $CB(L^p(M_1), L^p(M_2))$.

Existence of CB strongly non decomposable Schur multipliers

PROPOSTION [A.-K.]: Let I be an index set. Suppose $1 < p < \infty$. Let $\phi : I \times I \to \mathbb{C}$ be bounded. If $M_\phi : S^p_I \to S^p_I$ belongs to the closure $\overline{\mathrm{Dec}(S^p_I)}$ in $CB(S^p_I)$, then ϕ belongs to the closure of $\{$"$B(\ell^2_I)$-Schur mult. functions" $\}$ in $\ell^\infty(I \times I)$.

Proof: uses again complementation of S^p-Schur multipliers.

COROLLARY: The triangular truncation $S^p_{\mathbb{Z}} \to S^p_{\mathbb{Z}}$, $[x_{ij}] \mapsto [\delta_{i \leq j} x_{ij}]$ is CB strongly non decomposable.

Proof: By contraposition, use the known fact [Bennett 1977] that a bounded Schur multiplier M_ϕ on $B(\ell^2_{\mathbb{Z}})$ with existing limits $\lim_i \lim_j \phi_{ij}$ and $\lim_j \lim_i \phi_{ij}$ has equal limits.

Continuous Schur multipliers

DEFINITION: Let (Ω, μ) be a σ-finite measure space and $\phi : \Omega \times \Omega \to \mathbb{C}$ a measurable function. Let $S^p_\Omega = L^p(B(L^2(\Omega, \mu)), \mathrm{tr})$. A mapping $M_\phi : S^p_\Omega \to S^p_\Omega$ is called **(continuous) Schur multiplier** if it maps an element $x \in S^2_\Omega \cap S^p_\Omega$ identified with a function $(\omega_1, \omega_2) \mapsto x(\omega_1, \omega_2)$ in $L^2(\Omega \times \Omega)$, to $(\omega_1, \omega_2) \mapsto \phi(\omega_1, \omega_2)x(\omega_1, \omega_2)$.

PROPOSITION: There exists a linear mapping

$$P_\Omega : CB(S^p_I) \to CB(S^p_I)$$

with the following properties:

1. P_Ω takes its values in the completely bounded S^p_Ω-Schur multipliers.
2. P_Ω is contractive.
3. $P_\Omega(S) = S$ as soon as S is already a cb S^p-Schur multiplier.
4. $P_\Omega(S)$ is completely positive as soon as S is completely positive.

Proof Complementation Continuous Schur multipliers

ELEMENTS OF THE PROOF: Let $A_1, \ldots, A_n$ be mutually disjoint measurable subsets of Ω. Let $E_n : L^2(\Omega) \to L^2(\Omega),\ f \mapsto \sum_{k=1}^{n} \frac{1}{\mu(A_k)} \langle f, 1_{A_k} \rangle 1_{A_k}$ be the associated conditional expectation. Then we can use the complementation of discrete Schur multipliers $P_{I_n}(\Psi_n S \Phi_n)$ with $\Psi_n : S^p_\Omega \to S^p_{I_n}, x \mapsto E_n x E'_n$ and $\Phi_n : S^p_{I_n} \to S^p_\Omega, x \mapsto E'_n x E_n$. Use now a w^* accumulation argument to capture a limit point $P_\Omega(S) = \lim \Phi_n P_{I_n}(\Psi_n S \Phi_n) \Psi_n$.

Definition of Fourier multipliers I

Goal: define non-commutative Fourier multipliers.
Let $m : \mathbb{R} \to \mathbb{C}$ be a bounded measurable function. An L^p-Fourier multiplier on $\mathbb{R}$ is a mapping of the form

$$T_m f = \mathcal{F}^{-1}[m\hat{f}] = \int_{\mathbb{R}} m(s)\hat{f}(s)e^{is(\cdot)}ds$$

which extends boundedly to $L^p(\mathbb{R}) \to L^p(\mathbb{R})$. For $s \in \mathbb{R}$, consider $\chi_s : \; L^2(\mathbb{R}) \to L^2(\mathbb{R}), \; f \mapsto e^{is(\cdot)}f(\cdot)$, which is a unitary mapping.
We have $\chi_{s_1}\chi_{s_2} = \chi_{s_1+s_2}$, so

$$\chi : \mathbb{R} \to B(L^2(\mathbb{R})), \; s \mapsto \chi_s$$

is a group homomorphism with values in the unitaries.

Definition of Fourier multipliers II

Now replace in the above $\mathbb{R}$ by a locally compact group G (not necessarily abelian), equipped with left Haar measure. We put for $s \in G$

$$\lambda_s : L^2(G) \to L^2(G),\ f \mapsto f(s^{-1}\cdot).$$

Then $\lambda_{s_1}\lambda_{s_2} = \lambda_{s_1 s_2}$, so that $\lambda : G \to B(L^2(G))$ is a homomorphism. We set $M = \mathrm{VN}(G)$ the von Neumann algebra generated by $\{\lambda_s : s \in G\}$. Let now $m : G \to \mathbb{C}$ be a bounded measurable function. $\mathrm{VN}(G)$ is equipped with the functional $\tau(\lambda_g) = \delta_{ge}$, which extends to a trace if G is unimodular. For f belonging to a dense subset of $L^p(\mathrm{VN}(G))$, we can write $f = \int_G \hat{f}(s)\lambda_s ds$ for some bounded measurable function $\hat{f} : G \to \mathbb{C}$. An L^p-Fourier multiplier on G is a mapping of the form

$$T_m f = \int_G m(s)\hat{f}(s)\lambda_s ds$$

which extends to a bounded operator on $L^p(\mathrm{VN}(G))$.

Complementation of Fourier multipliers

THEOREM: Let G be a discrete group. For a completely bounded mapping $S : L^p(\mathrm{VN}(G)) \to L^p(\mathrm{VN}(G))$ let $m_S : G \to \mathbb{C},\ s \mapsto \tau(S(\lambda_s)\lambda_s^*)$. Then the linear mapping

$$P_G : CB(L^p(\mathrm{VN}(G))) \to CB(L^p(\mathrm{VN}(G))),\ S \mapsto T_{m_S}$$

has the following properties:

1. P_G takes its values in the completely bounded L^p-Fourier multipliers.
2. P_G is contractive.
3. $P_G(S) = S$ as soon as S is already a cb L^p-Fourier multiplier.
4. $P_G(S)$ is completely positive as soon as S is completely positive.

Application: Strongly non decomposable Fourier multipliers

QUESTION: Given a locally compact group G and $1 < p < \infty$, does there exist a CB strongly non decomposable Fourier multiplier on $L^p(\mathrm{VN}(G))$?

PROPOSITION [Arendt-Voigt 1991]: Let G be an abelian group. If $T_m : L^p(\mathrm{VN}(G)) \to L^p(\mathrm{VN}(G))$ belongs to $\overline{\mathrm{Dec}(L^p(\mathrm{VN}(G)))}$, then $m : G \to \mathbb{C}$ is continuous.

EXAMPLE [Arendt-Voigt 1991]: Let $G = \mathbb{R}$. Then the Fourier multiplier T_m with symbol $m(t) = \mathrm{sign}(t)$ is CB strongly non decomposable.

Strongly non decomposable Fourier multipliers

PROPOSITION [A.-K.]: Let G be a discrete group and $m : G \to \mathbb{C}$ a bounded measurable function.

1. Let $1 \leq p \leq \infty$. If $T_m \in \overline{\mathrm{Dec}(L^p(\mathrm{VN}(G)))}^{CB(L^p(\mathrm{VN}(G)))}$, then m belongs to the closure of the completely bounded L^∞-Fourier multipliers, closure in ℓ^∞_G.
2. If $\lim_n m(g^n)$ and $\lim_n m(g^{-n})$ exist for some $g \in G$, and m belongs to the above closure, then necessarily $\lim_n m(g^n) = \lim_n m(g^{-n})$.
3. Let $1 < p < \infty$, $n \in \mathbb{N}$ and $G = \mathbb{F}_n$ the free group of n generators. Then there exists a CB strongly non decomposable self-adjoint Fourier multiplier on $L^p(\mathrm{VN}(\mathbb{F}_n))$.

Proof of Proposition

PROOF: 1. Show that $\|T_m - R\|_{L^p(\mathrm{VN}(G))\to L^p(\mathrm{VN}(G))} \geq dist_{\ell^\infty_G}(m, \text{cb } L^\infty\text{-mult.})$ for any $R \in \mathrm{Dec}(L^p(\mathrm{VN}(G)))$: We can write $R = R_1 - R_2 + i(R_3 - R_4)$ with R_j completely positive. By the preceding Theorem, $P_G(R_j)$ is a completely positive Fourier multiplier, so $P_G(R)$ is a Fourier multiplier belonging to $\mathrm{Dec}(L^p(\mathrm{VN}(G)))$.
$\Longrightarrow P_G(R) \in \mathrm{Dec}(\mathrm{VN}(G)) \subseteq CB(\mathrm{VN}(G))$. We deduce

$$\begin{aligned}\|T_m - R\|_{CB(L^p(\mathrm{VN}(G)))} &\geq \|P_G(T_m) - P_G(R)\|_{cb,p\to p} \\ &\geq \|T_m - P_G(R)\|_{2\to 2} \\ &\geq d_{\ell^\infty_G}(T_m, \text{cb } L^\infty\text{-mult.})\end{aligned}$$

Proof of Proposition

2. Assume first T_m is a cb. L^∞ Fourier multiplier.
Then according to [Bozejko Fendler 1984] there exist a Hilbert space H and $P, Q : G \to H$ with $\sup_{x \in G} \|P(x)\|_H, \|Q(x)\|_H < \infty$ such that $m(y^{-1}x) = \langle P(x), Q(y) \rangle_H$ for $x, y \in G$. $(P(g^i))_i$ and $(Q(g^j))_j$ are bounded sequences in H, thus admit w^* convergent subsequences. Then it follows that
$\lim_{i\to\infty} m(g^{i-j}) = \lim_{i\to\infty} m(g^i) = \lim_{i\to\infty} \langle P(g^i), Q(g^j) \rangle =$
$\langle \lim_k P(g^{i_k}), \lim_k P(g^{j_k}) \rangle$.
Similarly, $\lim_{j\to\infty} m(g^{i-j}) = \langle \lim_k P(g^{i_k}), \lim_k Q(g^{j_k}) \rangle$.
Thus, $\lim_n m(g^n) = \lim_n m(g^{-n})$.
If T_m belongs only to the ℓ^∞_G closure of the cb. L^∞ Fourier multipliers, use a simple approximation argument.

Proof of Proposition

3. We can choose the Fourier multiplier on $L^p(\mathrm{VN}(\mathbb{F}_n))$ a non-commutative Riesz transform from [Junge Mei Parcet 2014]: The symbol is $m(g) = \langle b(g), h \rangle_H / \sqrt{\psi(g)}$ for some representing real Hilbert space H, a "length function" $\psi : G \to \mathbb{R}_+$, and an affine representation $b : G \to H$, $b(g_{i_1}^{j_1} \ldots g_{i_N}^{j_N}) = j_1 h_{i_1} + \ldots + j_N h_{i_N}$. Since m is real valued, T_m is self-adjoint. Moreover, $m(g_1^n) = \mathrm{sign}(n)$, so that $\lim_n m(g_1^n) = 1 \neq -1 = \lim_n m(g_1^{-n})$. Thus, by parts 1. and 2., T_m is CB strongly non decomposable.

Complementation of Fourier multipliers for non-discrete groups

The complementation of Fourier multipliers $P_G : CB(L^p(\mathrm{VN}(G))) \to CB(L^p(\mathrm{VN}(G)))$ assumed that G is *discrete*. In fact, if G is discrete, the trace $\tau(\lambda_g) = \delta_{ge}$ is *finite*, so that the unitaries λ_g generating $\mathrm{VN}(G)$ belong to $L^p(\mathrm{VN}(G))$ and $\Delta : \mathrm{VN}(G) \to \mathrm{VN}(G)\overline{\otimes}\mathrm{VN}(G)$, $\lambda_g \mapsto \lambda_g \otimes \lambda_g$ extends naturally to a bounded operator on $L^p(\mathrm{VN}(G))$.

This breaks down if G is not discrete and $p \neq \infty$.

But: some non-discrete groups admit an approximation by discrete groups.

DEFINITION: [Caspers Parcet Perrin Ricard 2014]: Let G be a locally compact group. G is called ADS (approximable by discrete subgroups) if there exists a family of lattices $(\Gamma_j)_{j\geq 1}$ in G and associated fundamental domains $(X_j)_{j\geq 1}$ which form a neighborhood basis of the identity. In this case, G is unimodular.

Fourier multiplier complementation for ADS groups

THEOREM [A.-K.]: Let G be an amenable ADS group. Assume that the fundamental domains are symmetric, i.e. $\mu(X_j^{-1} \Delta X_j) = 0$, where μ is left Haar measure. Assume moreover that

$$\frac{1}{\mu(X_j)} \int_G \frac{\mu(X_j \cap \gamma X_j g)^2}{\mu(X_j)^2} d\mu(g) \to c \quad (j \to \infty) \tag{1}$$

for some $c > 0$, uniformly in $\gamma \in \Gamma_j$.
Then for $1 \leq p \leq \infty$ there exists a linear mapping

$$P_G : CB(L^p(\mathrm{VN}(G))) \to CB(L^p(\mathrm{VN}(G)))$$

of norm at most $\frac{1}{c}$ with the properties:

1. $P_G(T)$ is a Fourier multiplier.
2. If T is completely positive, then $P_G(T)$ is completely positive.
3. If $T = T_m$ is a Fourier multiplier on $L^p(\mathrm{VN}(G))$ with uniformly continuous symbol $m : G \to \mathbb{C}$, then $P_G(T_m) = T_m$.

Fourier multiplier complementation for ADS groups

PARTS OF THE PROOF: Use an approximation idea of [Caspers Parcet Perrin Ricard 2014]: Let $h_j = \int_{X_j} \lambda_s d\mu(s) \in \mathrm{VN}(G)$ and define for $1 \leq p \leq \infty$:

$$\Phi_j^p : L^p(\mathrm{VN}(\Gamma_j)) \to L^p(\mathrm{VN}(G)),\ \lambda_\gamma \mapsto \mu(X_j)^{-2+\frac{1}{p}} h_j^* \lambda_\gamma h_j.$$

Then Φ_j^p is completely positive and completely contractive for any $1 \leq p \leq \infty$. Further let $\Psi_j^p = \left(\Phi_j^{p^*}\right)^* : L^p(\mathrm{VN}(G)) \to L^p(\mathrm{VN}(\Gamma_j))$. Define now for given $T \in CB(L^p(\mathrm{VN}(G)))$ the Fourier multiplier on the discrete group Γ_j

$$T_{m_j} = \frac{1}{c} P_{\Gamma_j}(\Psi_j^p T \Phi_j^p).$$

Let further the mollified Fourier multiplier symbol on G : $\widetilde{m_j} = \frac{1}{\mu(X_j)} 1_{X_j} * (m_j \mu_{\Gamma_j}) * 1_{X_j}$. Now show that $T_{\widetilde{m_j}}$ converges to a CB L^p-Fourier multiplier on G and use (1) to show that $T_{\widetilde{m_j}}$ converges to T_m if $T = T_m$.

More on ADS groups

COROLLARY:

1. Let G_0 be a discrete amenable group and G_1 an LCA group which is ADS, satisfying (1) for some $c > 0$, for example $G_1 = \mathbb{R}^n$ with $c = (\frac{2}{3})^n$. Let G_0 act on G_1 via a suitable homomorphism $\phi : G_0 \to \mathrm{Aut}(G_1)$. Then the semidirect product $G = G_0 \ltimes_\phi G_1$ is amenable ADS and (1) holds. Consequently, the above Theorem applies.
2. Let G_1 be an LCA compactly generated Lie group, so isomorphic to $\mathbb{R}^n \times \mathbb{T}^m \times \mathbb{Z}^l \times F$ with F a finite abelian group. Then G_1 is ADS and satisfies (1). Moreover, for G_0 a subgroup of $S_n \times S_m$ there exists a nontrivial homomorphism $\phi : G_0 \to \mathrm{Aut}(G_1)$ as in 1., exchanging "axes" in G_1. Consequently, the Theorem applies.

OPEN QUESTION: Find non-abelian Lie groups satisfying (1).

Existence of strongly non regular Fourier multipliers

QUESTION: Let G be a locally compact *abelian* group and $1 < p < \infty$. Does there exist a **strongly non regular Fourier multiplier (snrFm)** on $L^p(G)$, i.e. a bounded L^p Fourier multiplier not belonging to $\overline{\mathrm{Dec}(L^p(G))}$?

Observations:

1. If G is finite, then a finite dimension argument shows that no strongly non regular Fourier multiplier can exist.
2. If $G = \mathbb{R}, \mathbb{Z}$ or $\mathbb{T}$, then by [Arendt-Voigt 1991], the Hilbert transform is an example of a strongly non regular Fourier multiplier on $L^p(G)$.
3. For LCA groups, $\mathrm{VN}(G) = L^\infty(\hat{G})$, where $\hat{G}$ is again an LCA group, the Pontryagin dual.

Structure Theorems of stongly non regular Fourier multipliers

IDEA: Try to pass from a snrFm on a subgroup/quotient group to a snrFm on the whole group. For $H \subseteq G$,
$H^{\perp} = \{\xi \in \hat{G} : \langle \xi, h \rangle = 1 \text{ for all } h \in H\}$.

PROPOSITION: Let G be a LCA group and H a *compact* subgroup of G. If $m : H^{\perp} \to \mathbb{C}$ is a complex function, we denote by $\tilde{m} : \hat{G} \to \mathbb{C}$ the extension of m which is zero off $H^{\perp}$. If T_m induces a snrFm $T_m : L^p(G/H) \to L^p(G/H)$, then $\tilde{m}$ induces a snrFm $T_{\tilde{m}} : L^p(G) \to L^p(G)$.

PARTS OF THE PROOF: Suppose that $T_{\tilde{m}}$ belongs to $\overline{\mathrm{Dec}(L^p(G))}$. Let $\epsilon > 0$. There exist some positive maps $R_j : L^p(G) \to L^p(G)$ and a bounded map $R : L^p(G) \to L^p(G)$ of norm $< \epsilon$ such that $T_{\tilde{m}} = R_1 - R_2 + i(R_3 - R_4) + R$. Using complementation we can assume that R_j and R are Fourier multipliers.
Show that R_j and R pass to the quotient group G/H.

Structure Theorems of strongly non regular Fourier multipliers

PROPOSITION: Let G be a LCA group and H be a closed subgroup of G. Denote $\pi : \hat{G} \to \hat{G}/H^{\perp}$ the canonical map. Let $m : \hat{H} \to \mathbb{C}$ be a complex function. Then $m \circ \pi : \hat{G} \to \mathbb{C}$ induces a snrFm $L^p(G) \to L^p(G)$ if and only if $m : \hat{H} \to \mathbb{C}$ induces a snrFm $L^p(H) \to L^p(H)$.

PROPOSITION: Let G be an infinite compact abelian group. Then there exists a snrFm $L^p(G) \to L^p(G)$, $1 < p < \infty$.

PARTS OF THE PROOF: G compact $\implies$ $\hat{G}$ discrete. If $\hat{G}$ contains an element of infinite order, then $\hat{G} \supseteq \mathbb{Z}$. Use [Arendt-Voigt 1991] and the above structure proposition to find a snrFm. Otherwise, $\hat{G}$ is torsion. Use abstract Paley-Littlewood multiplier theory to find a snrFm of the form $m = \sum_{n=0}^{\infty} 1_{Y_{2n+1}} - 1_{Y_{2n}}$, where $(Y_n)_n$ is an increasing exhaustive sequence of finite subgroups of $\hat{G}$.

Existence of strongly non regular Fourier multipliers

PROPOSITION: Let G be an infinite discrete abelian group. Then there exists a strongly non regular Fourier multiplier on $L^p(G)$, $1 < p < \infty$.

THEOREM: Let G be an infinite LCA group. Then there exists a strongly non regular Fourier multiplier on $L^p(G)$, $1 < p < \infty$.

PARTS OF THE PROOF: The General Structure Theorem for LCA groups says that G is isomorphic with $\mathbb{R}^n \times G_0$ with $n \geq 0$ and G_0 is an LCA group containing a compact subgroup K such that G_0/K is discrete.
Distinguish 3 cases:
1.) if $n \geq 1$, then use the Hilbert transform on $\mathbb{R}$ and the structure proposition above.
2.) If $n = 0$, then $G \cong G_0$. If K is infinite, then use the above proposition for infinite compact groups.
3.) If K is finite, then G_0 itself must be discrete, so use the above proposition for infinite discrete groups.

Property $(\mathcal{P})$

Let $T : M \to M$ be a w^* continuous operator. T is said to satisfy property $(\mathcal{P})$ if there exist $v_1, v_2 : M \to M$ such that

$$\begin{pmatrix} v_1 & T \\ T^\circ & v_2 \end{pmatrix} : M_2(M) \to M_2(M)$$

is completely positive, completely contractive and self-adjoint.

REMARK: If T satisfies property $(\mathcal{P})$, then T is contractively decomposable and self-adjoint, but the converse fails.

THEOREM [A.-K.]: Let $G = \mathbb{F}_n$ be the free group. Let $T_m : \mathrm{VN}(G) \to \mathrm{VN}(G)$ be a Fourier multiplier satisfying $(\mathcal{P})$. Let $1 < p < \infty$. Then T_m satisfies the noncommutative Matsaev's inequality. More precisely, for any complex polynomial P, we have

$$\|P(T_m)\|_{cb, L^p \to L^p} \leq \|P(S)\|_{cb, \ell^p_{\mathbb{Z}} \to \ell^p_{\mathbb{Z}}},$$

where S is the unitary right shift on $\ell^p_{\mathbb{Z}}$.

Property $(\mathcal{P})$

What are the Fourier multipliers satisfying property $(\mathcal{P})$?

CONJECTURE: If $T : \mathrm{VN}(\mathbb{F}_n) \to \mathrm{VN}(\mathbb{F}_n)$ is a block-radial Fourier multiplier, i.e. if $f = \lambda_{g_{i_1}^{k_1}} \dots \lambda_{g_{i_N}^{k_N}}$ with $i_1 \neq i_2 \neq \dots \neq i_N$, then $Tf = \phi(N)f$, then T satisfies $(\mathcal{P})$ if and only if T is completely contractive and ϕ is real-valued (i.e. T self-adjoint).

Thank you for your attention