De Finetti reductions

and parallel repetition of multi-player non-local games

joint work with Andreas Winter

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Outline

1. De Finetti type theorems
2. Multi-player non-local games
3. Using de Finetti reductions to study the parallel repetition of multi-player non-local games
4. Summary, open questions and perspectives

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Classical finite de Finetti Theorem (Diaconis/Freedman)

Let $P^{(n)}$ be an exchangeable p.d. in n r.v.'s, i.e. for any $\pi \in \mathcal{S}_n$, $P^{(n)} \circ \pi = P^{(n)}$.
For any $k \leqslant n$, denote by $P^{(k)}$ the marginal p.d. of $P^{(n)}$ in k r.v.'s.
Then, there exists a p.d. μ on the set of p.d.'s in 1 r.v. s.t. $\left\| P^{(k)} - \int_Q Q^{\otimes k} d\mu(Q) \right\|_1 \leqslant \frac{k^2}{n}$.

$\rightarrow$ The marginal p.d. (in a few variables) of an exchangeable p.d. is well-approximated by a convex combination of product p.d.'s.

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Quantum finite de Finetti Theorem (Christandl/König/Mitchison/Renner)

Let $\rho^{(n)}$ be a permutation-symmetric state on $(\mathbf{C}^d)^{\otimes n}$, i.e. for any $\pi \in \mathcal{S}_n$, $U_\pi \rho^{(n)} U_\pi^\dagger = \rho^{(n)}$.
For any $k \leqslant n$, denote by $\rho^{(k)} = \mathrm{Tr}_{(\mathbf{C}^d)^{\otimes n-k}} \rho^{(n)}$ the reduced state of $\rho^{(n)}$ on $(\mathbf{C}^d)^{\otimes k}$.
Then, there exists a p.d. μ on the set of states on $\mathbf{C}^d$ s.t. $\left\| \rho^{(k)} - \int_\sigma \sigma^{\otimes k} d\mu(\sigma) \right\|_1 \leqslant \frac{2kd^2}{n}$.

$\rightarrow$ The reduced state (on a few subsystems) of a permutation-symmetric state is well-approximated by a convex combination of product states.

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Let $\rho^{(n)}$ be a permutation-symmetric state on $(\mathbf{C}^d)^{\otimes n}$. Then,

$$\rho^{(n)} \leqslant (n+1)^{d^2-1} \int_\sigma \sigma^{\otimes n} d\mu(\sigma), \ \mu : \text{uniform p.d. over the set of states on } \mathbf{C}^d.$$

Canonical application : If f is an order-preserving linear form s.t. $f \leqslant \varepsilon$ on 1-particle states, then $f^{\otimes n} \leqslant \text{poly}(n)\varepsilon^n$ on permutation-symmetric n-particle states.

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Drawback : All permutation-symmetric states are upper-bounded by the same mixture of tensor power states $\rightarrow$ Any other information is lost...

"Flexible" de Finetti reduction for quantum states

Let $\rho^{(n)}$ be a permutation-symmetric state on $(\mathbf{C}^d)^{\otimes n}$. Then,

$$\rho^{(n)} \leqslant (n+1)^{3d^2-1} \int_\sigma F\left(\rho^{(n)}, \sigma^{\otimes n}\right)^2 \sigma^{\otimes n} d\mu(\sigma),\ \mu : \text{uniform p.d. over the set of states on } \mathbf{C}^d.$$

Fidelity between two states (or p.d.'s) ρ, σ : $F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1 \leqslant 1$ (equality iff $\rho = \sigma$).

What is the “flexible” de Finetti reduction good for ?

$$\rho^{(n)} \leqslant \text{poly}(n) \int_{\sigma} F\left(\rho^{(n)}, \sigma^{\otimes n}\right)^2 \sigma^{\otimes n} d\mu(\sigma)$$

State-dependent upper-bound : Amongst states of the form $\sigma^{\otimes n}$, only those which have a high fidelity with the state of interest $\rho^{(n)}$ are given an important weight.
$\rightarrow$ Useful when one knows that $\rho^{(n)}$ satisfies some additional property : only states $\sigma^{\otimes n}$ approximately satisfying this same property should have a non-negligible fidelity weight...

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One canonical example of application :

If $\mathcal{N}^{\otimes n}(\rho^{(n)}) = \rho^{(n)}$, for some quantum channel (completely positive and trace preserving map) $\mathcal{N}$, then there exists a p.d. $\widetilde{\mu}$ over the range of $\mathcal{N}$ s.t.

$$\rho^{(n)} \leqslant \mathrm{poly}(n) \int_{\sigma} F\left(\rho^{(n)}, \sigma^{\otimes n}\right)^2 \sigma^{\otimes n} d\widetilde{\mu}(\sigma).$$

$\rightarrow$ No weight on states $\sigma^{\otimes n}$ s.t. $\sigma \notin \mathrm{Range}(\mathcal{N})$.

Particular case : $\mathcal{N} : \sigma \mapsto \sum_x \langle x|\sigma|x\rangle |x\rangle\langle x| = \sum_x P_\sigma(x) |x\rangle\langle x|$ quantum-classical channel.
$\rightarrow$ If $\mathcal{X}$ is finite and $P^{(n)}$ is a permutation-invariant p.d. on $\mathcal{X}^n$, then there exists a p.d. $\widetilde{\mu}$ over the set of p.d.'s on $\mathcal{X}$ s.t. $P^{(n)} \leqslant \mathrm{poly}(n) \int_Q F\left(P^{(n)}, Q^{\otimes n}\right)^2 Q^{\otimes n} d\widetilde{\mu}(Q)$.

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ℓ-player non-local games

ℓ cooperating but separated players. Each player i receives an input $x_i \in \mathcal{X}_i$ ("query") and produces an output $a_i \in \mathcal{A}_i$ ("answer"). They win if some binary predicate $V(a_1, \ldots, a_\ell, x_1, \ldots, x_\ell)$ on the answers & queries is satisfied. To achieve this, they can agree on a joint strategy before the game starts, but then cannot communicate anymore.

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Description of an ℓ-player non-local game G

- **Input alphabet** : $\underline{\mathcal{X}} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_\ell$. **Output alphabet** : $\underline{\mathcal{A}} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_\ell$. (both finite)
- **Game distribution** $=$ P.d. on $\underline{\mathcal{X}}$: $\{T(\underline{x}) \in [0,1],\ \underline{x} \in \underline{\mathcal{X}}\}$.
- **Game predicate** $=$ Binary predicate on $\underline{\mathcal{A}} \times \underline{\mathcal{X}}$: $\{V(\underline{a},\underline{x}) \in \{0,1\},\ (\underline{a},\underline{x}) \in \underline{\mathcal{A}} \times \underline{\mathcal{X}}\}$.
- **Players' strategy** $=$ Conditional p.d. on $\underline{\mathcal{A}}$ given $\underline{\mathcal{X}}$: $\{P(\underline{a}|\underline{x}) \in [0,1],\ (\underline{a},\underline{x}) \in \underline{\mathcal{A}} \times \underline{\mathcal{X}}\}$.

$\rightarrow$ Belongs to a set of "allowed strategies", depending on the kind of correlation resources that the players have (e.g. shared randomness, quantum entanglement, no-signalling boxes etc.)

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Value of a game G over a set of allowed strategies $AS(\underline{\mathcal{A}}|\underline{\mathcal{X}})$

Maximum winning probability for players playing G with strategies $P \in AS(\underline{\mathcal{A}}|\underline{\mathcal{X}})$:

$$\omega_{AS}(G) = \max \left\{ \sum_{\underline{a}\in\underline{\mathcal{A}},\underline{x}\in\underline{\mathcal{X}}} T(\underline{x}) V(\underline{a},\underline{x}) P(\underline{a}|\underline{x}) \ : \ P \in AS(\underline{\mathcal{A}}|\underline{\mathcal{X}}) \right\}$$

$\rightarrow$ Bell functional of particular form : all coefficients in $[0,1]$

- **Classical correlations** : $P \in \mathrm{C}(\underline{\mathcal{A}}|\underline{\mathcal{X}})$ if

$$\forall\ \underline{x} \in \underline{\mathcal{X}},\ \forall\ \underline{a} \in \underline{\mathcal{A}},\ P(\underline{a}|\underline{x}) = \sum_{m \in \mathcal{M}} Q(m) P_1(a_1|x_1\, m) \cdots P_\ell(a_\ell|x_\ell\, m),$$

for some p.d. Q on $\mathcal{M}$ and some p.d.'s $P_i(\cdot|x_i\, m)$ on $\mathcal{A}_i$.

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- **Quantum correlations** : $P \in \mathrm{Q}(\underline{\mathcal{A}}|\underline{\mathcal{X}})$ if

$$\forall\, \underline{x} \in \underline{\mathcal{X}},\ \forall\, \underline{a} \in \underline{\mathcal{A}},\ P(\underline{a}|\underline{x}) = \langle \psi | M(x_1)_{a_1} \otimes \cdots \otimes M(x_\ell)_{a_\ell} | \psi \rangle,$$

  for some state $|\psi\rangle$ on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_\ell$ and some POVMs $M(x_i)$ on $\mathcal{H}_i$.

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- **No-signalling correlations** : $P \in NS(\underline{\mathcal{A}}|\underline{\mathcal{X}})$ if

$$\forall\, I \subsetneq [\ell],\ \forall\, \underline{x} \in \underline{\mathcal{X}},\ \forall\, a_I \in \mathcal{A}_I,\ P(a_I|\underline{x}) = Q(a_I|x_I),$$

  for some p.d.'s $Q(\cdot|x_I)$ on $\mathcal{A}_I$.

- **Sub-no-signalling correlations** : $P \in SNOS(\underline{\mathcal{A}}|\underline{\mathcal{X}})$ if

$$\forall\, I \subsetneq [\ell],\ \forall\, \underline{x} \in \underline{\mathcal{X}},\ \forall\, a_I \in \mathcal{A}_I,\ P(a_I|\underline{x}) \leqslant Q(a_I|x_I),$$

  for some p.d.'s $Q(\cdot|x_I)$ on $\mathcal{A}_I$.

Remark : To check that a conditional p.d. P is NS, it is enough to check that it satisfies the NS conditions on subsets of the form $I = [\ell] \setminus \{i\}$, i.e. that for each $1 \leqslant i \leqslant \ell$, the marginal of P on $\underline{\mathcal{A}} \setminus \mathcal{A}_i | \underline{\mathcal{X}}$ does not depend on $\mathcal{X}_i$. But this is probably false for SNOS.

Some remarks on no-signalling and sub-no-signalling correlations

Players sharing (sub-)no-signalling correlations : no limitation is assumed on their physical power, apart from the fact that they cannot signal information instantaneously from one another. In the no-signalling case, players are forced to always produce an output, whatever input they received, while in the sub-no-signalling case they are even allowed to abstain from doing so.

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Relating the NS and the SNOS values of games

- Clearly, for any game G, $\omega_{NS}(G) \leqslant \omega_{SNOS}(G)$. And there are examples of games G s.t. $\omega_{SNOS}(G) = 1$ while $\omega_{NS}(G) < 1$ (e.g. anti-correlation game).
- If G is a 2-player game, then $\omega_{NS}(G) = \omega_{SNOS}(G)$ (reason : for any 2-party *SNOS* correlation, there exists a 2-party *NS* correlation dominating it pointwise).
- If G is an ℓ-player game whose distribution T has full support, then $\omega_{NS}(G) < 1 \ \Rightarrow \ \omega_{SNOS}(G) < 1$ (more quantitatively : $\omega_{SNOS}(G) \geqslant 1 - \delta \ \Rightarrow \ \omega_{NS}(G) \geqslant 1 - \Gamma\delta$, where $\Gamma > 1$ only depends on T).

The ℓ players play n instances of G in parallel : Each player i receives its n inputs $x_i^{(1)},\ldots,x_i^{(n)} \in \mathcal{X}_i$ together and produces its n outputs $a_i^{(1)},\ldots,a_i^{(n)} \in \mathcal{A}_i$ together.
Product game distribution on $\underline{\mathcal{X}}^n$: $T^{\otimes n}\left(\underline{x}^n\right) = T\left(\underline{x}^{(1)}\right)\cdots T\left(\underline{x}^{(n)}\right)$.

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Game $\mathbf{G^n}$: The players win if they win all n instances of G.
$\rightarrow$ Product game predicate on $\underline{\mathcal{A}}^n \times \underline{\mathcal{X}}^n$: $V^{\otimes n}\left(\underline{a}^n,\underline{x}^n\right) = V\left(\underline{a}^{(1)},\underline{x}^{(1)}\right)\cdots V\left(\underline{a}^{(n)},\underline{x}^{(n)}\right)$.

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Game $\mathbf{G^{t/n}}$: The players win if they win any t (or more) instances of G amongst the n.
$\rightarrow$ Game predicate on $\underline{\mathcal{A}}^n \times \underline{\mathcal{X}}^n$ defined as : $V^{t/n}\left(\underline{a}^n,\underline{x}^n\right) = 1$ if $\sum_{i=1}^n V\left(\underline{a}^{(i)},\underline{x}^{(i)}\right) \geqslant t$ and $V^{t/n}\left(\underline{a}^n,\underline{x}^n\right) = 0$ otherwise. In particular : $G^{n/n} = G^n$.

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The value $\omega_{AS}(G^n)$, resp. $\omega_{AS}(G^{t/n})$, is the maximum winning probability for players playing G^n, resp. $G^{t/n}$, with strategies $P \in AS(\underline{\mathcal{A}}^n | \underline{\mathcal{X}}^n)$.

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Question : For AS being either C, Q, NS or $SNOS$, we clearly have

$$\omega_{AS}(G)^n \leqslant \omega_{AS}(G^n) \leqslant \omega_{AS}(G).$$

But in the case where $\omega_{AS}(G) < 1$, what is the true behavior of $\omega_{AS}(G^n)$? Does it decay to 0 exponentially (in n), and if so at which rate ? More generally, does $\omega_{AS}(G^{t/n})$ as well decay to 0 exponentially as soon as $t/n > \omega_{AS}(G)$?

Intuitively, why should de Finetti reductions be useful to understand the parallel repetition of multi-player games ?

Observation : Obviously, the game distribution $T_{\underline{X}}^{\otimes n}$ and the game predicate $V_{\underline{AX}}^{\otimes n}$ of G^n are both permutation-invariant.

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Consequence : One can assume w.l.o.g. that the optimal winning strategy $P_{\underline{\mathcal{A}}^n|\underline{\mathcal{X}}^n}$, in the set of allowed strategies $AS(\underline{\mathcal{A}}^n|\underline{\mathcal{X}}^n)$, for G^n is permutation-invariant as well. And hence,

$$T_{\underline{\mathcal{X}}}^{\otimes n} P_{\underline{\mathcal{A}}^n|\underline{\mathcal{X}}^n} \leqslant \mathrm{poly}(n) \int_{Q_{\underline{\mathcal{A}}\underline{\mathcal{X}}}} F\left(T_{\underline{\mathcal{X}}}^{\otimes n} P_{\underline{\mathcal{A}}^n|\underline{\mathcal{X}}^n}, Q_{\underline{\mathcal{A}}\underline{\mathcal{X}}}^{\otimes n} \right)^2 Q_{\underline{\mathcal{A}}\underline{\mathcal{X}}}^{\otimes n} \, dQ_{\underline{\mathcal{A}}\underline{\mathcal{X}}}.$$

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$$T_{\underline{\mathcal{X}}}^{\otimes n} P_{\underline{\mathcal{A}}^n|\underline{\mathcal{X}}^n} \leqslant \mathrm{poly}(n) \int_{Q_{\underline{\mathcal{A}\mathcal{X}}}} F\left(T_{\underline{\mathcal{X}}}^{\otimes n} P_{\underline{\mathcal{A}}^n|\underline{\mathcal{X}}^n}, Q_{\underline{\mathcal{A}\mathcal{X}}}^{\otimes n} \right)^2 Q_{\underline{\mathcal{A}\mathcal{X}}}^{\otimes n} \, dQ_{\underline{\mathcal{A}\mathcal{X}}}.$$

Goal : Show that the only p.d.'s $Q_{\underline{\mathcal{A}\mathcal{X}}}^{\otimes n}$ for which the fidelity weight is not exponentially small are those s.t. $Q_{\underline{\mathcal{A}\mathcal{X}}}$ is close to being of the form $T_{\underline{\mathcal{X}}} R_{\underline{\mathcal{A}}|\underline{\mathcal{X}}}$ with $R_{\underline{\mathcal{A}}|\underline{\mathcal{X}}} \in AS(\underline{\mathcal{A}}|\underline{\mathcal{X}})$. Because what happens when playing G^n with such strategy $R_{\underline{\mathcal{A}}|\underline{\mathcal{X}}}^{\otimes n}$ is trivially understood.

Outline

Parallel repetition of sub-no-signalling ℓ-player games

Let G be an ℓ-player game s.t. $\omega_{SNOS}(G) \leqslant 1-\delta$ for some $0<\delta<1$. Then, for any $n\in\mathbb{N}$ and $t \geqslant (1-\delta+\alpha)n$, $\omega_{SNOS}(G^n) \leqslant \left(1-\delta^2/5C_\ell^2\right)^n$ and $\omega_{SNOS}(G^{t/n}) \leqslant \exp\left(-n\alpha^2/5C_\ell^2\right)$, where $C_\ell = 2^{\ell+1}-3$.

Parallel repetition of (sub-)no-signalling multi-player games : some results

Parallel repetition of sub-no-signalling ℓ-player games

Let G be an ℓ-player game s.t. $\omega_{SNOS}(G) \leqslant 1-\delta$ for some $0<\delta<1$. Then, for any $n \in \mathbb{N}$ and $t \geqslant (1-\delta+\alpha)n$, $\omega_{SNOS}(G^n) \leqslant \left(1-\delta^2/5C_\ell^2\right)^n$ and $\omega_{SNOS}(G^{t/n}) \leqslant \exp\left(-n\alpha^2/5C_\ell^2\right)$, where $C_\ell = 2^{\ell+1}-3$.

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Parallel repetition of no-signalling ℓ-player games with full support

Let G be an ℓ-player game whose input distribution T has full support, and s.t. $\omega_{NS}(G) \leqslant 1-\delta$ for some $0<\delta<1$. Then, for any $n \in \mathbb{N}$ and $t \geqslant (1-\delta+\alpha)n$, $\omega_{NS}(G^n) \leqslant \left(1-\delta^2/5C_\ell^2\Gamma^2\right)^n$ and $\omega_{NS}(G^{t/n}) \leqslant \exp\left(-n\alpha^2/5C_\ell^2\Gamma^2\right)$, where $C_\ell = 2^{\ell+1}-3$ and Γ is a constant which only depends on T.

Starting point : The optimal winning strategy $P_{\underline{\mathcal{A}}^n|\underline{\mathcal{X}}^n} \in SNOS(\underline{\mathcal{A}}^n|\underline{\mathcal{X}}^n)$ for G^n satisfies

$$T_{\underline{\mathcal{X}}}^{\otimes n} P_{\underline{\mathcal{A}}^n|\underline{\mathcal{X}}^n} \leqslant \text{poly}(n) \int_{Q_{\underline{\mathcal{A}\mathcal{X}}}} \widetilde{F}(Q_{\underline{\mathcal{A}\mathcal{X}}})^{2n} Q_{\underline{\mathcal{A}\mathcal{X}}}^{\otimes n} \, dQ_{\underline{\mathcal{A}\mathcal{X}}},$$

where $\widetilde{F}\left(Q_{\underline{\mathcal{A}\mathcal{X}}}\right) = \min_{\emptyset \neq I \subsetneq [\ell]} \max_{R_{\mathcal{A}_I|\mathcal{X}_I}} F\left(T_{\underline{\mathcal{X}}} R_{\mathcal{A}_I|\mathcal{X}_I}, Q_{\mathcal{A}_I\underline{\mathcal{X}}}\right)$.

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Separating the "very-signalling" and the "not-too-signalling" parts in the integral :

Fix $0 < \varepsilon < 1$ and define $\mathcal{P}_\varepsilon = \left\{ Q_{\underline{\mathcal{A}\mathcal{X}}} \ : \ \max_{\emptyset \neq I \subsetneq [\ell]} \min_{R_{\mathcal{A}_I|\mathcal{X}_I}} \frac{1}{2} \| T_{\underline{\mathcal{X}}} R_{\mathcal{A}_I|\mathcal{X}_I} - Q_{\mathcal{A}_I \underline{\mathcal{X}}} \|_1 \leqslant \varepsilon \right\}$.

- $Q_{\underline{\mathcal{A}\mathcal{X}}} \notin \mathcal{P}_\varepsilon \ \Rightarrow \ \widetilde{F}\left(Q_{\underline{\mathcal{A}\mathcal{X}}}\right)^2 \leqslant 1 - \varepsilon^2$.
- $Q_{\underline{\mathcal{A}\mathcal{X}}} \in \mathcal{P}_\varepsilon \ \Rightarrow \ \exists\, R_{\underline{\mathcal{A}}|\underline{\mathcal{X}}} \in SNOS(\underline{\mathcal{A}}|\underline{\mathcal{X}}) : \ \frac{1}{2} \| T_{\underline{\mathcal{X}}} R_{\underline{\mathcal{A}}|\underline{\mathcal{X}}} - Q_{\underline{\mathcal{A}\mathcal{X}}} \|_1 \leqslant C_\ell \varepsilon$.

$\rightarrow$ Technical lemma behind : If a conditional p.d. approximately satisfies each of the NS constraints, up to an error ε, then it is $C\varepsilon$-close to an exact SNOS p.d.

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Putting everything together : The winning probability when playing G^n with strategy $P_{\underline{\mathcal{A}}^n|\underline{\mathcal{X}}^n}$ is upper-bounded by $\mathrm{poly}(n)\left((1-\varepsilon^2)^n + (1-\delta+2C_\ell\varepsilon)^n\right)$.

It then just remains to choose $\varepsilon = C_\ell\left(\left(1+\delta/C_\ell^2\right)^{1/2} - 1\right)$ and get rid of the polynomial pre-factor in order to conclude.

Outline

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- If ℓ players sharing sub-no-signalling correlations have a probability at most $1-\delta$ of winning a game G, then their probability of winning a fraction at least $1-\delta+\alpha$ of n instances of G played in parallel is at most $\exp(-nc_\ell\alpha^2)$, where $c_\ell > 0$ is a constant which depends only on ℓ.
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  $\rightarrow$ Optimal dependence in α, even in the special case $\alpha = \delta$.
- In the case $\ell = 2$, this is equivalent to the analogous concentration result for the no-signalling value of G (cf. Holenstein).
- In the case where the distribution of G has full support, this implies a similar concentration result for the no-signalling value of G, but with a highly game-dependent constant in the exponent (cf. Buhrman/Fehr/Schaffner and Arnon-Friedman/Renner/Vidick).
  $\rightarrow$ What about games where some of the potential queries are never asked to the players ?

Broader perspectives

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- **Classical case** : Exponential decay of $\omega(G^n)$ and $\omega(G^{t/n})$ under parallel repetition for any 2-player game G (Raz, Holenstein, Rao).
  Quantum case : Exponential decay of $\omega(G^n)$ under parallel repetition for any 2-player game G with full support (Chailloux/Scarpa).

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  "Standard" proof techniques : Show that conditioned on the event "the players have already won k instances of the game", the probability is high that :
  i) they lose in at least 1 of the $n-k$ remaining instances $\rightarrow$ exponential decay of $\omega(G^n)$,
  ii) they lose in most of the $n-k$ remaining instances $\rightarrow$ exponential decay of $\omega(G^{t/n})$.
  But not so straightforward and not easily generalizable to more than 2 players...
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  $\rightarrow$ What about tackling the problem via de Finetti reductions ? Obstacle : classical and quantum conditions cannot be read off on the marginals...
- Using flexible de Finetti reductions to prove the (weakly) multiplicative or additive behavior of certain quantities appearing in QIT (e.g. output norms or entropies of quantum channels) : work in progress...

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