On some properties of the Unitary Dual Group

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Definition (Dual group)

A (unital) dual group $(\mathcal{A},\Delta,\delta,\Sigma)$ is a (unital) $*$-algebra $\mathcal{A}$, and three (unital) $*$-homomorphisms $\Delta:\mathcal{A}\to\mathcal{A}\sqcup\mathcal{A}$, $\delta:\mathcal{A}\to\mathbb{C}$ and $\Sigma:\mathcal{A}\to\mathcal{A}$, such that

- *The map Δ is a coassociative coproduct:* $(Id\bigsqcup\Delta)\circ\Delta=(\Delta\bigsqcup Id)\circ\Delta$
- *The map δ is a counit:* $(\delta\bigsqcup Id)\circ\Delta=Id=(Id\bigsqcup\delta)\circ\Delta$
- *The map Σ is a coinverse:* $(\Sigma\sqcup Id)\circ\Delta=1_{\mathcal{A}}\circ\delta=(Id\sqcup\Sigma)\circ\Delta.$

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Remark

- Free product "more noncommutative" than tensor product
- Unlike compact quantum groups, we have a counit and a coinverse (cf. comapct quantum groups of Kac type)

This is the dual group we will be interested in the sequel

Definition

Let $n \geq 0$. We call unitary dual group the following dual group: $(U\langle n\rangle, \Delta, \delta, \Sigma)$ where:

- *The unital $*$-algebra $U\langle n\rangle$ is generated by n^2 elements $(u_{ij})_{1\leq i,j\leq n}$ with the relations*

$$\sum_k u_{ki}^* u_{kj} = \delta_{ij} = \sum_k u_{ik} u_{jk}^*.$$

- *The coproduct is given on the generators by $\Delta(u_{ij}) = \sum_k u_{ik}^{(1)} u_{kj}^{(2)}$.*
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The relations defining $U\langle n\rangle$ can be summed up by saying that $u = (u_{ij})_{1\leq i,j\leq n}$ is a unitary matrix in $\mathcal{M}_n(U\langle n\rangle)$. Unless the quantum case, we do not suppose that $\bar{u} = (u_{ij}^*)_{1\leq i,j\leq n}$ is unitary!

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Going further

Let $(U_t^{(d)})_{t\geq 0}$ be a Brownian motion on the (classical) unitary group $U(nd)$ (with n fixed). Then, we define the following process:

$$\begin{aligned} j_t^{(d)} \quad : \quad & U\langle n\rangle \to L^{\infty-} \otimes \mathcal{M}_d \\ & u_{ij} \mapsto [U_t^{(d)}]_{ij} \end{aligned}$$

where $[.]_{ij}$ designates the (i,j) block of size $d \times d$ of a matrix.

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Where $L^{\infty-}$ are the random variable having moments of all order.
We say that $j_t^{(d)}$ is a quantum random variable on $U\langle n\rangle$.

Theorem (U, 2014; second proof Cebron, U, 2015)

When d goes to infinity, the process $(j_t^{(d)})_{t\geq 0}$ tends towards a free Lévy process $(j_t)_{t\geq 0}$ on $U\langle n\rangle$.

Remark

When $n = 1$, this is a classical result obtained by Biane in 1997.

Theorem (U, 2014)

The limit free Lévy process verifies

$$j_t : u_{ij} \mapsto \Psi_{ij}$$

where Ψ_t is a $n \times n$ matrix whose elements verify the free stochastic differential equation

$$d\Psi = \frac{i}{\sqrt{n}} dX_t \Psi_t - \frac{1}{2}\Psi_t dt$$

with X_t being a self-conjugate matrix whose elements are free additive Brownian motions (free equivalent of the GUE).

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$$dU_t = idH_t U_t - \frac{1}{2} U_t dt$$

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Theorem (U, 2014)

The process $(j_t)_{t\geq 0}$ verifies a property called gaussianity (in the sense of Schürmann).

Remark

It is thus a good candidate to be called a Brownian motion

Definition (Cebron, U, 2015)

We define a Haar state on $U\langle n\rangle$ as being a state h such that for any state ϕ:

$$(h \odot \phi)\Delta = h = (\phi \odot h)\Delta$$

with $\odot$ being one of the five notions of independances (free, tensor, monotone, antimonotone, boolean).

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Remark

The boolean, monotone and antimonotone independances are defined on non-unital algebras. For our definition to make sense, we must consider $U\langle n\rangle = \mathbb{C} \oplus Ker\delta$.

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Theorem (Cebron, U, 2015)

For $n \geq 2$, there exists no tensor, free, boolean, monotone or anti-monotone Haar state.
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Proof.

The ingredient of the proof is to build specific states ϕ and to find a contradiction... □

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Theorem (Cebron, U, 2015)

For $n \geq 2$, there exists no Haar trace for the boolean, monotone and anti-monotones independances, but there exists one for the free one and one for the tensor one.

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We take (A, ψ) a noncommutative $*$-probability space. We define:

$$\tilde{A} = E_{11}[A \sqcup M_n(\mathbb{C})]E_{11}$$

and

$$h = n[\psi * tr]$$

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We also define the map

$$j : \begin{array}{ccc} U\langle n\rangle & \rightarrow & \tilde{A} \\ u_{ij} & \mapsto & E_{1i}UE_{j1} \end{array}$$

with $U \in A$ a Haar unitary.

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The state $h \circ j$ is a free Haar trace on $U\langle n\rangle$.

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Proof.

Ingredients: Theorem of Nica and Speicher about compressed algebra
cumulants of Haar unitary
considering partitions.

□

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We consider the Hilbert space $H = l^2(\mathbb{Z}) \otimes \bigotimes_{k\in\mathbb{Z}} \mathcal{M}_n(\mathbb{C})$, where the number of matrices different of I_n is finite.
We define the operator:

$$\begin{aligned} & U_{ij}(\delta_k \otimes (\ldots \otimes M_{k-1} \otimes M_k \otimes M_{k+1} \otimes \ldots)) \\ = \; & \delta_{k+1} \otimes (\ldots \otimes M_{k-1} \otimes E_{ji} M_k \otimes M_{k+1} \otimes \ldots) \\ & U_{ij}^*(\delta_k \otimes (\ldots \otimes M_{k-1} \otimes M_k \otimes M_{k+1} \otimes \ldots)) \\ = \; & \delta_{k-1} \otimes (\ldots \otimes E_{ij} M_{k-1} \otimes M_k \otimes M_{k+1} \otimes \ldots) \end{aligned}$$

And we set:

$$\Omega = \delta_0 \otimes \bigotimes_{k\in\mathbb{Z}} I_n$$

and $j : u_{ij} \mapsto U_{ij}$.

Theorem (Cebron, U, 2015)

The state $h(a) = < \Omega, j(a)\Omega >$ *is the tensor Haar trace.*

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The state $h(a) = < \Omega, j(a)\Omega >$ *is the tensor Haar trace.*

Proof.

It is easy to compute the values of h on words alternating in $*$. To compute more general words,we use

$$S_k = \left\{ l \in \{1, \ldots, r\} : k = \sharp\{m > l : \epsilon_m = \emptyset\} - \sharp\{m \geq l : \epsilon_m = *\} \right\}$$

and h then factorizes according to these S_k. □

Theorem (Cebron, U, 2015)

The free Haar trace is faithful, whereas the tensor one is not.

results already obtained:

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- Replacing U by a free Lévy process U_t in our method, we get unitary quantum Lévy processes. We can write their Schürmann triple.

Question in process:

- What does it mean that we get only a Haar traces?
- What happens with other dual groups?

- Construction of a free Lévy process as high-dimensional limit of a Brownian motion on the unitary group, M.Ulrich, Infin. Dim. Anal. Quant. Prob., 2015, vol. 18
- Haar states and Lévy processes on the unitary dual group, G.Cébron, M.Ulrich, ArXiv:1505.08083, to appear in JFA